

Non-simply-laced Clusters of Finite Type via Frobenius Morphism

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Abstract

By showing the compatibility of folding almost positive roots and folding cluster categories, we prove that there is a one-to-one correspondence between seeds and tilting seeds in non-simply-laced finite cases.

Key words: Frobenius morphism, cluster, cluster category, mutation.

1 Introduction

Introduced by Fomin and Zelevinsky [FZ], cluster algebras are a family of commutative algebras generated by the so-called cluster variables. In [FZ2] the authors classify all cluster algebras of finite type, and construct for all finite types a one-to-one correspondence between almost positive roots and cluster variables. It is pointed out that this classification coincides with the classification of Cartan matrices. Then [MRZ] [BMRRT] (and independently [CCS]) developed the representation theoretical approach to study cluster algebras. [BMR] [CC] [CK] [CK2] [Hub2] [Z] are papers following this approach. In particular, they establish a one-to-one correspondence between clusters and tilting objects in the so-called cluster category.

In [MRZ] the authors claim that non-simply-laced clusters can be studied by folding the corresponding simply-laced ones. Using this method Dupont [D] realizes non-simply-laced cluster algebras as quotients of simply-laced cluster algebras. Deng and Du [DD] establish a

link between representations of quivers over $\overline{\mathbb{F}}_q$ (simply-laced case) and representations of \mathbb{F}_q -species (non-simply-laced case). This link is generalized in [DD2] to the homotopy category, the bounded derived category and the root category. In this paper we extend it further to the cluster category. Namely, we study folding cluster categories. Applying this and the study of folding almost positive roots to known results on simply-laced clusters of finite type, we provide a new approach to understand non-simply-laced clusters of finite type. The main result is

Theorem 1.1. *For non-simply-laced finite type, there is a one-to-one correspondence between clusters and tilting objects in the cluster category. Under this correspondence, mutation of seeds is exactly mutation of tilting seeds.*

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2 Preliminaries

We first introduce some notations.

For a category \mathcal{E} , denote by $ind\mathcal{E}$ the set of isomorphism classes of indecomposable objects of \mathcal{E} . Depending on the context, it also means a representative set. For an object X of \mathcal{E} , denote by $[X]$ the isomorphism class of X .

Let A be an algebra. By an A -module, we mean a finite dimensional left A -module. Denote by $modA$ the category of A -modules.

Let q be a prime power, \mathbb{F}_q the finite field with q elements, and $\overline{\mathbb{F}}_q$ a fixed algebraic closure of \mathbb{F}_q .

2.1 Frobenius morphisms

In [DD] [DD2] [DD3], Deng and Du study relation between the representation theory of an $\overline{\mathbb{F}}_q$ -algebra A with a Frobenius automorphism F and that of the algebra A^F of fixed points, which is an \mathbb{F}_q -algebra. This subsection is devoted to a brief introduction to their results.

A *Frobenius map* F on a $\overline{\mathbb{F}}_q$ -space V is a map from V to itself satisfying

- (i) $F(\lambda v) = \lambda^q v$, $\forall \lambda \in \overline{\mathbb{F}}_q$, $v \in V$.
- (ii) $\forall v \in V$, $\exists n \in \mathbb{N}$, s.t. $F^n(v) = v$.

Lemma 2.1. (i) Let V be a finite dimensional $\overline{\mathbb{F}}_q$ -space with a Frobenius map F_V , then $V^F = \{v \in V \mid F_V(v) = v\}$ is a finite dimensional \mathbb{F}_q -space such that $V = \overline{\mathbb{F}}_q \otimes_{\mathbb{F}_q} V^F$. Moreover, the \mathbb{F}_q -dimension of V^F equals the $\overline{\mathbb{F}}_q$ -dimension of V .

(ii) Let V, W be two finite dimensional $\overline{\mathbb{F}}_q$ -spaces with Frobenius maps F_V and F_W respectively, then $F : f \mapsto F_W \circ f \circ F_V^{-1}$ is a Frobenius map on $\text{Hom}_{\overline{\mathbb{F}}_q}(V, W)$, and $\text{Hom}_{\mathbb{F}_q}(V^F, W^F) = (\text{Hom}_{\overline{\mathbb{F}}_q}(V, W))^F$.

In the rest of this subsection we assume A is a finite dimensional algebra over $\overline{\mathbb{F}}_q$ with a fixed Frobenius morphism. By definition a *Frobenius morphism* F on A is a Frobenius map on the vector space A and also an automorphism of \mathbb{F}_q -algebras. Note that the algebra A^F of fixed points is a finite dimensional algebra over \mathbb{F}_q , and $\overline{\mathbb{F}}_q \otimes A^F = A$. Moreover, A is hereditary if and only if A^F is hereditary.

For an A -module M and $r \in \mathbb{Z}$, define an A -module $M^{[r]} = A \otimes_{F^r} M$, where $a \otimes m = 1 \otimes F^{-r}(a)m$. If we write $m^{(r)}$ for $1 \otimes m$ and write the new action by \cdot_r , then explicitly $a \cdot_r m^{(r)} = (F^{-r}(a)m)^{(r)}$. For an A -module homomorphism $f : M \rightarrow N$, we define an A -module homomorphism $f^{[r]} : M^{[r]} \rightarrow N^{[r]}$, $m^{(r)} \mapsto (f(m))^{(r)}$. In this way, $(\cdot)^{[r]}$ is an autoequivalence of $\text{mod}A$ and $(\cdot)^{[r]}(\cdot)^{[s]} = (\cdot)^{[r+s]}$. In particular, $(\cdot)^{[r]}$ commutes with the Auslander-Reiten translation τ .

The selfequivalence $(\cdot)^{[r]}$ of the Abelian category $\text{mod}A$ extends to $\mathcal{C}(A)$, the category of complexes of A -modules, which commutes with the shift functor [1]. Precisely, for $M^\cdot = \{M_i, d_i\}$, we define $\{M_i, d_i\}^{[r]} = \{M_i^{[r]}, d_i^{[r]}\}$, or equivalently, $M^{\cdot[r]} = A \otimes_{F^r} M^\cdot$. For $f^\cdot = \{f_i\} : M^\cdot \rightarrow N^\cdot$ a homomorphism of complexes, $f^{\cdot[r]} = \{f_i^{[r]}\} : M^{\cdot[r]} \rightarrow N^{\cdot[r]}$ is a homomorphism of complexes too. Moreover, f is homotopic to zero (resp. a quasi-isomorphism) if and only if so is $f^{[r]}$. Therefore $(\cdot)^{[r]}$ induces an equivalence of triangulated categories $(\cdot)^{[r]} : \mathcal{C}(A) \rightarrow \mathcal{C}(A)$ where $\mathcal{C}(A)$ is the bounded homotopy category $\mathcal{K}^b(A)$ or the bounded derived category $\mathcal{D}^b(A)$.

In the following let $C(A) = \text{mod}A, \mathcal{C}^b(A), \mathcal{K}^b(A)$, or $\mathcal{D}^b(A)$. We have $\dim_{\overline{\mathbb{F}}_q} \text{Hom}_{\mathcal{C}(A)}(M, N) = \dim_{\overline{\mathbb{F}}_q} \text{Hom}_{\mathcal{C}(A)}(M^{[1]}, N^{[1]})$. A pair (M, ϕ_M) is called an *F -stable object* in $\mathcal{C}(A)$ if $\phi_M : M^{[1]} \rightarrow M$ is an isomorphism in $\mathcal{C}(A)$. We also say that M is an F -stable object. Denote by $C^F(A)$ the category of F -stable objects whose morphisms $f : (M, \phi_M) \rightarrow (N, \phi_N)$ are morphisms $f : M \rightarrow N$ in $\mathcal{C}(A)$ satisfying $f \circ \phi_M = \phi_N \circ f^{[1]}$. On $C^F(A)$ we have the Auslander-Reiten translation τ defined by $\tau(M, \phi_M) = (\tau_M, \tau\phi_M)$.

Let (M, ϕ_M) be an F -stable A -module, i.e. an F -stable object in $\text{mod}A$. Then $F_M : M \rightarrow M$, $m \mapsto \phi_M(m^{(1)})$ is a Frobenius map on M which is compatible with the A -module structure of M , i.e. $F_M(am) = F(a)F_M(m)$ for any $a \in A, m \in M$. We see that M^F is an

A^F -module. An morphism $f : (M, \phi_M) \rightarrow (N, \phi_N)$ between F -stable A -modules induces a morphism $f^F = f|_{M^F} : M^F \rightarrow N^F$ between A^F -modules. Thus we have a functor Φ from $C^F(A)$ to $C(A^F)$. For an object (M, ϕ_M) in $C^F(A)$ we will denote by M^F its image under Φ . We have

Theorem 2.2. (i) *Φ is an equivalence (of abelian categories and triangulated categories respectively) from $C^F(A)$ to $C(A^F)$ with inverse functor $\overline{\mathbb{F}}_q \otimes_{\mathbb{F}_q} -$. Moreover, this equivalence commutes with the Auslander-Reiten translation.*

(ii) *Let (M, ϕ_M) and (N, ϕ_N) be two objects in $C^F(A)$. Then M^F and N^F are isomorphic in $C(A^F)$ if and only if M and N are isomorphic in $C(A)$. Moreover, $\text{Hom}_{C(A^F)}(M^F, N^F) = (\text{Hom}_{C(A)}(M, N))^F$.*

It is shown that for each object M in $C(A)$ there exists $r \in \mathbb{N}$ such that $M^{[r]} \cong M$. For such r the module $M \oplus M^{[1]} \dots \oplus M^{[r-1]}$ is F -stable. If r is minimal such that $M^{[r]} \cong M$ (such r is called the F -period of M) then $\tilde{M} = M \oplus M^{[1]} \dots \oplus M^{[r-1]}$ is indecomposable in $C^F(A)$.

2.2 Clusters

In this subsection we follow [FZ2] [FZ3]. Let Γ be a valued graph of finite type with vertex set I , denote by $\Phi(\Gamma)$ the root system and by $\Phi_{\geq -1}(\Gamma)$ the set of almost positive roots (i.e. the set of positive roots and negative simple roots). A *cluster variable* is an element in $\Phi_{\geq -1}(\Gamma)$.

Let s_i be the simple reflection of the Weyl group of $\Phi(\Gamma)$ corresponding to $i \in I$, and let σ_i be the permutation of $\Phi_{\geq -1}(\Gamma)$ defined as follows

$$\sigma_i(\alpha) = \begin{cases} \alpha & \text{if } \alpha = -\alpha_j, j \neq i \\ s_i(\alpha) & \text{otherwise} \end{cases}$$

Let $I = I^+ \cup I^-$ be a partition of the vertex set I into completely disconnected subsets. Let $\tau_+ = \prod_{i \in I^+} \sigma_i$ and $\tau_- = \prod_{i \in I^-} \sigma_i$. Define the compatibility degree $(\alpha || \beta)_\Gamma = (\alpha || \beta)$ by $(-\alpha_i || \beta) =$ the coefficient of α_i in β , where α_i is a simple root and β is a positive root, and extend τ_\pm -invariantly to $(,) : \Phi_{\geq -1}(\Gamma) \times \Phi_{\geq -1}(\Gamma) \rightarrow \mathbb{N}_0$. If Γ is simply-laced then $(\alpha || \beta) = (\beta || \alpha)$.

We say that $\alpha, \beta \in \Phi_{\geq -1}(\Gamma)$ are *compatible* if $(\alpha || \beta) = 0$. A *cluster* is defined to be a maximal compatible subset of $\Phi_{\geq -1}(\Gamma)$. The cardinality of a cluster equals the cardinality of I . Two elements α, β ($\alpha \neq \beta$) in $\Phi_{\geq -1}(\Gamma)$ form an *exchange pair* if there is a cluster \underline{x} such that $\alpha \in \underline{x}$ and $(\underline{x} \setminus \{\alpha\}) \cup \{\beta\}$ is again a cluster.

Let $\underline{x} = \{x_i\}_{i \in I}$ a cluster, and $B = (b_{ij})$ be any matrix realization of the graph Γ , which is an anti-symmetrizable matrix. We call the pair (\underline{x}, B) an *initial seed*. For each $k \in I$, one can define the *mutation* along direction k which is a pair (\underline{x}', B') where $\underline{x}' = (\underline{x} \setminus x_k) \cup \{x'_k\}$ with x'_k defined via \underline{x} and B (in fact x_k and x'_k form an exchange pair), and $B' = (b'_{ij})$ is defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{b_{ik}|b_{ij}| + |b_{ik}|b_{kj}}{2} & \text{otherwise} \end{cases}$$

We write $(\underline{x}', B') = \mu_k(\underline{x}, B)$, and $B' = \mu_k B$. We call a pair (\underline{x}', B') a *seed* if \underline{x}' is a cluster and B' is an anti-symmetrizable matrix such that the pair is an iterated mutation of the initial seed.

To each anti-symmetrizable matrix $B = (b_{ij})$ with integer entries we associate a valued quiver Q_B . Precisely, if $b_{ij} > 0$, then Q_B has $i \xrightarrow{(-b_{ji}, b_{ij})} j$. Note that Q_B has no loops or oriented cycles of length 2. In fact this defines a bijection between the set of anti-symmetrizable matrices with integer entries and the set of valued quivers without loops or oriented cycles of length 2. In the following, we denote by B_Q the anti-symmetrizable matrix corresponding to a valued quiver Q .

2.3 Cluster category

Let A be a finite dimensional hereditary algebra. Let τ be the Auslander-Reiten translation of $\mathcal{D}^b(A)$ and let S denote the shift functor. Following [BMRRT] we form the orbit category $\mathcal{C}_A = \mathcal{D}^b(A)/\tau^{-1}S$. This is a Krull-Schmidt category. Precisely, the objects of \mathcal{C}_A are exactly the objects in $\mathcal{D}^b(A)$, and for $X, Y \in \mathcal{C}_A$, $\text{Hom}_{\mathcal{C}_A}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(A)}(X, (\tau^{-1}S)^i Y)$. It is shown in [K] that \mathcal{C}_A is a triangulated category with shift functor S , and the canonical functor from $\mathcal{D}^b(A)$ to \mathcal{C}_A is a triangle functor.

Define $\text{Ext}_{\mathcal{C}_A}^i(X, Y) = \text{Hom}_{\mathcal{C}_A}(X, S^i Y)$ for $i \in \mathbb{Z}$. Then $\text{Ext}_{\mathcal{C}_A}^1(X, Y) = \text{Ext}_{\mathcal{C}_A}^1(Y, X)$. An object T in \mathcal{C}_A is called *exceptional* if $\text{Ext}_{\mathcal{C}_A}^1(T, T) = 0$. An exceptional object T is called a *(basic cluster)-tilting object* (resp. *almost complete tilting object*) if T has exactly n (resp. $n-1$) pairwise non-isomorphic indecomposable direct summands, where n is the number of isoclasses of simple A -modules. An exceptional indecomposable object M is called a *complement* of an almost complete tilting object \overline{T} if $\overline{T} \oplus M$ is a tilting object. An almost complete tilting object has precisely two complements. Moreover, for two exceptional indecomposable objects M and M^* , there exists an almost complete tilting object \overline{T} such that M and M^* are exactly the two complements of \overline{T} if and only if $\dim_{D_M} \text{Ext}_{\mathcal{C}_A}^1(M, M^*) = 1 = \dim_{D_M^*} \text{Ext}_{\mathcal{C}_A}^1(M^*, M)$ where

D_M and D_{M^*} are the division algebras $\text{End}(M)/\text{rad}\text{End}(M)$ and $\text{End}(M^*)/\text{rad}\text{End}(M^*)$ respectively.

Let $T = T_1 \oplus \cdots \oplus T_n$ be a tilting object. The algebra $\text{End}_{\mathcal{C}_A}(T)^{\text{op}}$ is called a *cluster-tilted algebra*. By [BMR] Proposition 3.2 the valued quiver Q_T of $\text{End}_{\mathcal{C}_A}(T)^{\text{op}}$ has no loops or oriented cycles of length 2. Write $B_T = B_{Q_T}$. The pair (T, B_T) is called a *tilting seed*. For $k = 1, \dots, n$, there exists a unique indecomposable object T_k^* non-isomorphic to T_k such that $T' = T_1 \oplus \cdots \oplus T_{k-1} \oplus T_k^* \oplus T_{k+1} \oplus \cdots \oplus T_n$ is again a tilting object. Therefore $(T', B_{T'})$ is again a tilting seed, called a *mutation* of (T, B_T) along direction k . We write $(T', B_{T'}) = \delta_k(T, B_T)$, and $B_{T'} = \delta_k B_T$. Any tilting seed is an iterated mutation of the tilting seed (SA, B_{SA}) .

2.4 Simply-laced clusters of finite type

Let Q be a Dynkin quiver with underlying graph Γ (hence Γ is a simply-laced Dynkin graph of finite type). Denote by I the set of vertices of Q (and Γ).

Let P_i be the indecomposable projective module corresponding to the vertex i . Then $\text{ind}\mathcal{C}_A = \text{ind}(\text{mod}A) \cup \{[SP_i]\}_{i=1, \dots, n}$. We define the dimension vector of SP_i to be $\underline{\dim}(SP_i) = -\alpha_i$. By Gabriel's theorem (see [G]) taking dimension vector defines a bijective correspondence between the set $\{SP_i \mid i \in I\} \cup \text{ind mod}(\overline{\mathbb{F}}_q Q)$ and $\Phi_{\geq -1}(\Gamma)$, namely, between $\text{ind}\mathcal{C}(Q)$ and the set of cluster variables.

Theorem 2.3. ([BMRRT]) (i) For $\alpha, \beta \in \Phi_{\geq -1}(\Gamma)$, let M_α, M_β be the corresponding indecomposable object in $\mathcal{C}(Q)$, then $(\alpha \parallel \beta)_\Gamma = \dim_{\overline{\mathbb{F}}_q} \text{Ext}_{\mathcal{C}(Q)}^1(M_\alpha, M_\beta)$.

(ii) The bijective correspondence induces a bijective correspondence between the isoclasses of basic (cluster-) tilting objects in $\mathcal{C}(Q)$ and clusters.

(iii) Two elements α, β of $\Phi_{\geq -1}(\Gamma)$ form an exchange pair if and only if their compatibility degree is 1.

Theorem 2.3 (iii) is first proved by Fomin and Zelevinsky in [FZ2] 3.5, 4.4 for all Dynkin diagrams Γ of finite type. Later we will give an analogue of the proof in [BMRRT] for non-simply-laced finite types.

3 Non simply-laced clusters of finite type

3.1 Folding cluster categories

We call an pair (Q, σ) an *admissible quiver* if Q is a (finite) quiver, and σ an admissible automorphism of Q (see [L] 12.1.1 or [DD] Example 3.5). To an admissible quiver (Q, σ) ,

we associate an \mathbb{F}_q -species Q^σ (see [DD] Section 6 for detailed construction). Conversely, each \mathbb{F}_q -species is associated to some admissible quiver ([DD] Theorem 6.5). σ induces a Frobenius morphism F on the algebra $\overline{\mathbb{F}}_q Q$ with $(\overline{\mathbb{F}}_q Q)^F = \mathbb{F}_q Q^\sigma$. Let $\mathcal{D}^b(Q) = \mathcal{D}^b(\overline{\mathbb{F}}_q Q)$ and $\mathcal{D}^b(Q^\sigma) = \mathcal{D}^b(\mathbb{F}_q Q^\sigma)$, then by Section 2.1 we have $\mathcal{D}^b(Q)^F \simeq \mathcal{D}^b(Q^\sigma)$. Form cluster categories $\mathcal{C}_Q = \mathcal{C}_{\overline{\mathbb{F}}_q Q}$, and $\mathcal{C}_{Q^\sigma} = \mathcal{C}_{\mathbb{F}_q Q^\sigma}$. The selfequivalence $(\cdot)^{[r]} : \mathcal{D}^b(Q) \rightarrow \mathcal{D}^b(Q)$ is well-defined and commutes with the Auslander-Reiten translation τ and the shift functor S , and hence commutes with $\tau^{-1}S$. Therefore we obtain a selfequivalence of the cluster category \mathcal{C}_Q , also denoted by $(\cdot)^{[r]}$.

Lemma 3.1. *Let $r \in \mathbb{Z}$ and M be an object in \mathcal{C}_Q . Assume $\phi : M^{[r]} \rightarrow M$ is a morphism in \mathcal{C}_Q , then ϕ is an isomorphism in \mathcal{C}_Q if and only if it is an isomorphism in $\mathcal{D}^b(Q)$.*

Proof. Assume ϕ is an isomorphism in \mathcal{C}_Q , then ϕ is an isomorphism in $\mathcal{D}^b(Q)$ from $M^{[r]}$ to $(\tau^{-1}S)^i M$ for some $i \in \mathbb{Z}$. By a direct comparison on the dimensions of the homology group of the two complexes we deduce that $i = 0$. The converse is obvious. \square

A pair (M, ϕ_M) is called an *F-stable object* in \mathcal{C}_Q if $\phi_M : M^{[1]} \rightarrow M$ is an isomorphism in \mathcal{C}_Q , that is, an object in $\mathcal{D}^b(Q)^F$ by Lemma 3.1. We also say that M is an *F-stable object*. Denote by \mathcal{C}_Q^F the category of *F-stable objects* whose morphisms $f : (M, \phi_M) \rightarrow (N, \phi_N)$ are morphisms $f : M \rightarrow N$ in \mathcal{C}_Q satisfying $f \circ \phi_M = \phi_N \circ f^{[1]}$.

Theorem 3.2. (i) \mathcal{C}_Q^F is equivalent to \mathcal{C}_{Q^σ} as triangulated categories. For an object (M, ϕ_M) in \mathcal{C}_Q^F we will denote by M^F the corresponding object in \mathcal{C}_{Q^σ} .

(ii) Let (M, ϕ_M) and (N, ϕ_N) be two objects in \mathcal{C}_Q^F . Then M^F and N^F are isomorphic in \mathcal{C}_{Q^σ} if and only if M and N are isomorphic in \mathcal{C}_Q . Moreover, $\text{Hom}_{\mathcal{C}_{Q^\sigma}}(M^F, N^F) = (\text{Hom}_{\mathcal{C}_Q}(M, N))^F$.

Proof. (i) Since $\tau^{-1}S$ commutes with $(\cdot)^{[1]}$, it induces a selfequivalence of $\mathcal{D}^b(Q)^F$ as a triangulated category. Moreover, let $(M, \phi_M), (N, \phi_N)$ be two objects in \mathcal{C}_Q^F (i.e. in $\mathcal{D}^b(Q)^F$) and $\xi : (M, \phi_M) \rightarrow (N, \phi_N)$ a morphism in \mathcal{C}_Q^F , i.e. $\xi : M \rightarrow N$ is a morphism in \mathcal{C}_Q or equivalently $\xi : M \rightarrow \bigoplus_i (\tau^{-1}S)^i N$ is a morphism in $\mathcal{D}^b(Q)$ and the following diagram is commutative in $\mathcal{D}^b(Q)$.

$$\begin{array}{ccc} M^{[1]} & \xrightarrow{\phi_M} & M \\ \downarrow \xi^{[1]} & & \downarrow \xi \\ \bigoplus (\tau^{-1}S)^i N^{[1]} & \xrightarrow{\bigoplus (\tau^{-1}S)^i \phi_N} & \bigoplus (\tau^{-1}S)^i N \end{array}$$

Therefore ξ is a morphism in \mathcal{C}_Q^F if and only if it is a morphism in $\mathcal{D}^b(Q)^F$ from (M, ϕ_M) to $(\bigoplus (\tau^{-1}S)^i N, \bigoplus (\tau^{-1}S)^i \phi_N)$, i.e. it is a morphism from (M, ϕ_M) to (N, ϕ_N) in $\mathcal{D}^b(Q)^F / (\tau^{-1}S)$.

Thus we have proved that \mathcal{C}_Q^F is canonically equivalent to $\mathcal{D}^b(Q)^F / \tau^{-1}S$.

Now by Theorem 2.2 it follows that $\mathcal{D}^b(Q)^F/(\tau^{-1}S)$ is equivalent to $\mathcal{D}^b(Q^\sigma)/(\tau^{-1}S) = \mathcal{C}_Q^F$, and we are done. \square

By the above proof the equivalence in Theorem 3.2 and its inverse can be constructed explicitly, but here we do not give the details. By Lemma 3.1 the following is a consequence of the corresponding result for the derived category $\mathcal{D}^b(Q)$.

Proposition 3.3. *For an object M in \mathcal{C}_Q there exists $r \in \mathbb{N}$ such that $M^{[r]} \cong M$.*

As in the derived category, for an r as in Proposition 3.3, the object $M \oplus M^{[1]} \oplus \dots \oplus M^{[r-1]}$ is F -stable. The minimal such r is called the F -period of M . We remark that the F -periods of an $\overline{\mathbb{F}}_q Q$ -module in $\text{mod}\overline{\mathbb{F}}_q Q$, $\mathcal{D}^b(Q)$, and \mathcal{C}_Q all coincide.

Proposition 3.4. *Assume σ of order t . Let M be an exceptional indecomposable $\overline{\mathbb{F}}_q Q$ -module, then the F -period of M is a divisor of t , i.e. $M^{[t]} \cong M$. Consequently, $(\cdot)^{[1]}$ induces a σ -action on $\text{ex.ind}(\mathcal{C}_Q)$, the set of isoclasses of exceptional indecomposable objects in \mathcal{C}_Q .*

Proof. Let $\{p_j\}_{j \in J}$ be the set of paths of Q , then it is a basis of $\overline{\mathbb{F}}_q Q$ stable under F^t . By Ringel [R], there exists a basis $\{m_l\}_{l \in L}$ of M such that $p_j m_l = \sum \lambda_{jll'} m_{l'}$ where $\lambda_{jll'} = 1, 0$ for all j, l, l' . Therefore $M^{[t]} \cong M$. \square

Proposition 3.5. *Let M be indecomposable object in \mathcal{C}_Q with F -period r , then $\tilde{M} = M \oplus M^{[1]} \dots \oplus M^{[r-1]}$ is indecomposable in \mathcal{C}_Q^F . Moreover, $D_{\tilde{M}^F} = \mathbb{F}_{q^r}$.*

Proof. Note that there exists $m \in \mathbb{Z}$ such that $S^m M$ is an object in $\text{mod}\overline{\mathbb{F}}_q Q$. Since S^m is a selfequivalence of \mathcal{C}_Q , we have that $S^m M$ is also of F -period r , and $D_{\tilde{M}^F} \cong D_{(\widetilde{S^m M})^F}$. Therefore we may assume M is an object in $\text{mod}\overline{\mathbb{F}}_q Q$. Then

$$D_{\tilde{M}^F} = \text{End}_{\mathcal{C}_Q^\sigma}(\tilde{M}^F)/\text{radEnd}_{\mathcal{C}_Q^\sigma}(\tilde{M}^F) = \text{End}_{\mathbb{F}_q Q^\sigma}(\tilde{M}^F)/\text{radEnd}_{\mathbb{F}_q Q^\sigma}(\tilde{M}^F) = \mathbb{F}_{q^r}.$$

The last equality follows from [DD] Theorem 5.1. \square

Theorem 3.6. (i) *Let M be an F -stable object in \mathcal{C}_Q . Then M^F is a tilting object in \mathcal{C}_Q^σ if and only if M is a tilting object in \mathcal{C}_Q .*

(ii) *Let M be an tilting object in \mathcal{C}_Q which is F -stable, then the associated Frobenius map on M induces a Frobenius morphism on the algebra $\text{End}_{\mathcal{C}_Q}(M)$. Moreover, $\text{End}_{\mathcal{C}_Q^\sigma}(M^F) = \text{End}_{\mathcal{C}_Q}(M)^F$ (therefore the theory introduced in Section 2.1 applies to cluster-tilted algebras). In particular F induces an admissible automorphism σ_M of Q_M such that Q_{M^F} is the underlying valued quiver of the \mathbb{F}_q -species $Q_M^{\sigma_M}$.*

Proof. This is because $\text{Ext}_{\mathcal{C}_{Q^\sigma}}^i(M^F, M^F) = \text{Ext}_{\mathcal{C}_Q}^i(M, M)^F$, where $i = 0, 1$. \square

Let $I = \{1, \dots, n\}$ be the vertex set of Q , then $I^\sigma = \{\tilde{i} | i \in I\}$ is the vertex set of Q^σ where \tilde{i} is the σ -orbit of $i \in I$. Let $T = T_1 \oplus \dots \oplus T_n$ be a tilting object in \mathcal{C}_Q , and is F -stable. Let $\bigoplus_{k' \in \tilde{k}} T_{k'}$ be an F -stable direct summand, indecomposable in \mathcal{C}_Q^F . Then there is a unique set $\{T_{k'}\}_{k' \in \tilde{k}}$ such that $T' = \bigoplus_{i \notin \tilde{k}} T_i \oplus \bigoplus_{k' \in \tilde{k}} T'_{k'}$ is a tilting object in \mathcal{C}_Q , and is F -stable. Write $B = B_{T^F} = (b_{ij})_{\tilde{i}, \tilde{j} \in I^\sigma}$, $B' = B_{T'^F} = (b'_{ij})_{\tilde{i}, \tilde{j} \in I^\sigma}$, $A = B_T = (a_{ij})_{i, j \in I}$, and $A' = B_{T'} = (a'_{ij})_{i, j \in I}$, then $A' = \prod_{k' \in \tilde{k}} \delta_{k'} A$, and $B' = \delta_{\tilde{k}} B$. By Theorem 3.6(ii) we have $b_{\tilde{i}\tilde{j}} = \sum_{j' \in \tilde{j}} a_{ij'}$, and $b'_{\tilde{i}\tilde{j}} = \sum_{j' \in \tilde{j}} a'_{ij'}$. We want a description of B' via B . The following proposition is proved in [D].

Proposition 3.7. $b'_{\tilde{i}\tilde{j}} = \begin{cases} -b_{\tilde{i}\tilde{j}} & \text{if } \tilde{i} = \tilde{k}, \text{ or } \tilde{j} = \tilde{k}, \\ b_{\tilde{i}\tilde{j}} + \frac{b_{\tilde{i}k}|b_{\tilde{k}\tilde{j}}| + |b_{\tilde{i}k}|b_{\tilde{k}\tilde{j}}}{2} & \text{otherwise.} \end{cases}$

Here we give a new proof. Write $\delta_k(A) = A^{(k)} = (a_{ij}^{(k)})$, then we have

Lemma 3.8. ([BMR]) $a_{ij}^{(k)} = \begin{cases} -a_{ij} & \text{if } i = k \text{ or } j = k, \\ a_{ij} + \frac{a_{ik}|a_{kj}| + |a_{ik}|a_{kj}}{2} & \text{otherwise.} \end{cases}$

From this formula we have the following

Lemma 3.9. Let $l_1, \dots, l_t \in \{1, \dots, n\}$ be t pairwise non adjacent vertices, i.e. $a_{l_p l_{p'}} = 0$ for any $p, p' \in \{1, \dots, t\}$. Write $\delta_{l_t} \cdots \delta_{l_1}(A) = A^{(l_1 \cdots l_t)} = (a_{ij}^{(l_1 \cdots l_t)})$, then

$$a_{ij}^{(l_1 \cdots l_t)} = \begin{cases} -a_{ij} & \text{if one of } i, j \in \{l_1, \dots, l_t\}, \text{ the other not,} \\ a_{ij} + \sum_{p=1}^t \frac{a_{il_p}|a_{l_p j}| + |a_{il_p}|a_{l_p j}}{2} & \text{otherwise.} \end{cases}$$

Proof. We prove by induction on t .

If $t = 1$, we have by Lemma 3.8

$$a_{ij}^{(l_1)} = \begin{cases} -a_{ij} & \text{if one of } i, j \in \{l_1\}, \text{ the other not,} \\ a_{ij} + \frac{a_{il_1}|a_{l_1 j}| + |a_{il_1}|a_{l_1 j}}{2} & \text{otherwise.} \end{cases}$$

Suppose the statement is true for $t - 1$. Then by Lemma 3.8

$$a_{ij}^{(l_1 \cdots l_t)} = \begin{cases} -a_{ij}^{(l_1 \cdots l_{t-1})} & \text{if one of } i, j \in \{l_t\}, \text{ the other not,} \\ a_{ij}^{(l_1 \cdots l_{t-1})} + \frac{a_{il_t}^{(l_1 \cdots l_{t-1})}|a_{l_t j}^{(l_1 \cdots l_{t-1})}| + |a_{il_t}^{(l_1 \cdots l_{t-1})}|a_{l_t j}^{(l_1 \cdots l_{t-1})}}{2} & \text{otherwise.} \end{cases}$$

We will use the following two conditions : (i) one of $i, j \in \{l_1, \dots, l_{t-1}\}$, the other not; (ii) one of $i, j \in \{l_t\}$, the other not.

Case 1 : condition (i) (ii) both hold. Namely, one of $i, j \in \{l_1, \dots, l_{t-1}\}$, the other $\notin \{l_1, \dots, l_{t-1}, l_t\}$. Then

$$a_{ij}^{(l_1 \dots l_t)} = -a_{ij}^{(l_1 \dots l_{t-1})} = a_{ij} = a_{ij} + \sum_{i=1}^p \frac{a_{il_p}|a_{lpj}| + |a_{il_p}|a_{lpj}}{2}.$$

Case 2 : condition (i) holds, but condition (ii) fails. Then if $i \in \{l_1, \dots, l_{t-1}\}$, $j \notin \{l_1, \dots, l_{t-1}\}$, then $a_{il_t}^{(l_1 \dots l_{t-1})} = -a_{il_t} = 0$; if $i \notin \{l_1, \dots, l_{t-1}\}$, $j \in \{l_1, \dots, l_{t-1}\}$, then $a_{il_t}^{(l_1 \dots l_{t-1})} = -a_{lj} = 0$. Therefore

$$a_{ij}^{(l_1 \dots l_t)} = a_{ij}^{(l_1 \dots l_{t-1})} + \frac{a_{il_t}^{(l_1 \dots l_{t-1})}|a_{lj}| + |a_{il_t}^{(l_1 \dots l_{t-1})}|a_{lj}}{2} = -a_{ij}.$$

Case 3 : condition (i) fails but condition (i) holds. Namely, one of $i, j \in \{l_t\}$, the other $\notin \{l_1, \dots, l_{t-1}, l_t\}$. Then

$$a_{ij}^{(l_1 \dots l_t)} = -a_{ij}^{(l_1 \dots l_{t-1})} = -(a_{ij} + \sum_{p=1}^{t-1} \frac{a_{il_p}|a_{lpj}| + |a_{il_p}|a_{lpj}}{2}) = -a_{ij}.$$

where the last equality is because either $a_{il_p} = 0$ or $a_{lpj} = 0$.

Case 4 : condition (i) (ii) both fail. Then

$$a_{ij}^{(l_1 \dots l_t)} = a_{ij}^{(l_1 \dots l_{t-1})} + \frac{a_{il_t}^{(l_1 \dots l_{t-1})}|a_{lj}| + |a_{il_t}^{(l_1 \dots l_{t-1})}|a_{lj}}{2}.$$

By induction hypothesis we have

$$\begin{aligned} a_{ij}^{(l_1 \dots l_{t-1})} &= a_{ij} + \sum_{p=1}^{t-1} \frac{a_{il_p}|a_{lpj}| + |a_{il_p}|a_{lpj}}{2}, \\ a_{il_t}^{(l_1 \dots l_{t-1})} &= a_{il_t} + \sum_{p=1}^{t-1} \frac{a_{il_p}|a_{lpj}| + |a_{il_p}|a_{lpj}}{2} = a_{il_t}, \\ a_{lj}^{(l_1 \dots l_{t-1})} &= a_{lj} + \sum_{p=1}^{t-1} \frac{a_{lj}|a_{lpj}| + |a_{lj}|a_{lpj}}{2} = a_{lj}. \end{aligned}$$

Therefore

$$a_{ij}^{(l_1 \dots l_t)} = a_{ij} + \sum_{p=1}^{t-1} \frac{a_{il_p}|a_{lpj}| + |a_{il_p}|a_{lpj}}{2} + \frac{a_{il_t}|a_{lj}| + |a_{il_t}|a_{lj}}{2} = a_{ij} + \sum_{p=1}^t \frac{a_{il_p}|a_{lpj}| + |a_{il_p}|a_{lpj}}{2}.$$

To conclude we get the desired result. \square

Proof of Proposition 3.7 : Since \tilde{k} is a σ -orbit of vertices it follows that vertices in \tilde{k} are pairwise non adjacent in A . The fact that Q_{TF} has no loops or oriented cycles of length 2 implies that for $m, l \in I$ all numbers in $\{a_{m'l'}\}_{m' \in \tilde{m}, l' \in \tilde{l}}$ has the same sign. Applying Lemma 3.9 we prove the desired result case by case.

Case 1 : $\tilde{i} = \tilde{k}, \tilde{j} = \tilde{k}$. Then $b'_{\tilde{i}\tilde{j}} = \sum_{j' \in \tilde{j}} a'_{ij'} = 0$.

Case 2 : $\tilde{i} = \tilde{k}, \tilde{j} \neq \tilde{k}$, or $\tilde{i} \neq \tilde{k}, \tilde{j} = \tilde{k}$. Namely, $i \in \{k_1, \dots, k_s\}$, $j \notin \{k_1, \dots, k_s\}$ or $i \notin \{k_1, \dots, k_s\}$, $j \in \{k_1, \dots, k_s\}$. Then $b'_{\tilde{i}\tilde{j}} = \sum_{j' \in \tilde{j}} a'_{ij'} = \sum_{j' \in \tilde{j}} a_{ij'}^{(k_1 \dots k_s)} = \sum_{j' \in \tilde{j}} (-a_{ij'}) = -b_{\tilde{i}\tilde{j}}$.

Case 3 : otherwise. In this case,

$$b'_{\tilde{i}\tilde{j}} = \sum_{j' \in \tilde{j}} a'_{ij'} = \sum_{j' \in \tilde{j}} (a_{ij'} + \sum_{k' \in \tilde{k}} \frac{a_{ik'}|a_{k'j}| + |a_{ik'}|a_{k'j}}{2}) = b_{\tilde{i}\tilde{j}} + \sum_{k' \in \tilde{k}} \frac{a_{ik'} \sum_{j' \in \tilde{j}} |a_{k'j'}| + |a_{ik'}| \sum_{j' \in \tilde{j}} a_{k'j'}}{2}.$$

For a fixed $k' \in \tilde{k}$ all numbers $a_{k'j'}$, $j' \in \tilde{j}$ have the same sign. Therefore

$$\begin{aligned} b'_{\tilde{i}\tilde{j}} &= b_{\tilde{i}\tilde{j}} + \sum_{k' \in \tilde{k}} \frac{a_{ik'} \sum_{j' \in \tilde{j}} a_{k'j'} + |a_{ik'}| \sum_{j' \in \tilde{j}} a_{k'j'}}{2} \\ &= b_{\tilde{i}\tilde{j}} + \sum_{k' \in \tilde{k}} \frac{a_{ik'}|b_{\tilde{k}\tilde{j}}| + |a_{ik'}|b_{\tilde{k}\tilde{j}}}{2} = b_{\tilde{i}\tilde{j}} + \frac{(\sum_{k' \in \tilde{k}} a_{ik'})|b_{\tilde{k}\tilde{j}}| + \sum_{k' \in \tilde{k}} |a_{ik'}|b_{\tilde{k}\tilde{j}}}{2}. \end{aligned}$$

We see that all numbers $a_{ik'}$, $k' \in \tilde{k}$ have the same sign. Therefore

$$b'_{\tilde{i}\tilde{j}} = b_{\tilde{i}\tilde{j}} + \frac{(\sum_{k' \in \tilde{k}} a_{ik'})|b_{\tilde{k}\tilde{j}}| + \sum_{k' \in \tilde{k}} a_{ik'}|b_{\tilde{k}\tilde{j}}|}{2} = b_{\tilde{i}\tilde{j}} + \frac{b_{\tilde{i}\tilde{k}}|b_{\tilde{k}\tilde{j}}| + |b_{\tilde{i}\tilde{k}}|b_{\tilde{k}\tilde{j}}}{2}.$$

This completes the proof. \square

3.2 Folding almost positive roots

Folding almost positive roots is a consequence of folding root systems, which is a strong tool in studying non-simply-laced Kac theorem, see [DD] [DX] [Hua] [Hub].

Let Q be a Dynkin quiver and σ an admissible automorphism. Then the associated \mathbb{F}_q -species Q^σ is the natural \mathbb{F}_q -species of certain valued quiver, which we also denote by Q^σ . Let Γ be the underlying diagram of Q , then σ induces an automorphism of Γ , which we also denote by σ . The associated valued graph Γ^σ is exactly the underlying graph of Q^σ . We recall that the vertex set of Q^σ (and Γ^σ) is $I^\sigma = \{\tilde{i} \mid i \in I\}$, where I is the vertex set of Q (and Γ) and \tilde{i} is the σ -orbit of $i \in I$.

σ induces an automorphism of $\Phi(\Gamma)$ and $\Phi_{\geq -1}(\Gamma)$ is a σ -stable subset. For $\alpha \in \Phi(\Gamma)$ of σ -period d we define $\tilde{\alpha} = \sum_{s=0}^{d-1} \sigma^s \alpha$. Then $\Phi(\Gamma)^\sigma = \{\tilde{\alpha} \mid \alpha \in \Phi(\Gamma)\} = \Phi(\Gamma^\sigma)$ and $\Phi_{\geq -1}(\Gamma)^\sigma = \{\tilde{\alpha} \mid \alpha \in \Phi_{\geq -1}(\Gamma)\} = \Phi_{\geq -1}(\Gamma^\sigma)$.

Let $I^\sigma = (I^\sigma)^+ \cup (I^\sigma)^-$ be a partition of I^σ into completely disconnected subsets then there is a partition $I = I^+ \cup I^-$ of I with I^+ and I^- both completely disconnected and σ -stable, and $(I^\pm)^\sigma = (I^\sigma)^\pm$. Define the compatibility degrees $(\parallel)_\Gamma$ and $(\parallel)_{\Gamma^\sigma}$ via these two partitions respectively, then

Lemma 3.10. $(\tilde{\alpha} \parallel \tilde{\beta})_{\Gamma^\sigma} = \sum_{t=0}^{d(\beta)-1} (\alpha \parallel \sigma^t \beta)_\Gamma$, where $d(\beta)$ is the σ -period of β .

Proof. This is because $(-\tilde{\alpha}_i \parallel \tilde{\beta})_{\Gamma^\sigma} = \sum_{t=0}^{d(\beta)-1} (-\alpha_i \parallel \sigma^t \beta)_\Gamma$ for any $i \in I$. \square

3.3 Non-simply-laced cluster of finite type

Let Q be a Dynkin quiver and σ an admissible automorphism. By Proposition 3.4 the self-equivalence $(\cdot)^{[1]}$ induces a σ -action on $\text{ind}\mathcal{C}_Q$. The following lemma shows that this action commutes with taking dimension vectors.

Lemma 3.11. *For any object M in \mathcal{C}_Q we have $\underline{\dim}(M^{[1]}) = \sigma(\underline{\dim} M)$.*

Proof. It suffices to prove for $M = SP_i$ where P_i is the indecomposable projective corresponding to the vertex $i \in I$, and for M an $\overline{\mathbb{F}}_q Q$ -module. By the comment after [DD] Proposition 4.2, we deduce that $P_i^{[1]} \cong P_{\sigma(i)}$. Therefore

$$\underline{\dim}((SP_i)^{[1]}) = \underline{\dim}(S(P_i)^{[1]}) = \underline{\dim}(SP_{\sigma(i)}) = -\alpha_{\sigma(i)} = \sigma(-\alpha_i) = \sigma(\underline{\dim}(SP_i)).$$

For M an $\overline{\mathbb{F}}_q Q$ -module, we have

$$\begin{aligned} \underline{\dim}(M^{[1]}) &= \sum_{i \in I} \dim_{\overline{\mathbb{F}}_q} \text{Hom}_Q(P_i, M^{[1]}) \alpha_i = \sum_{i \in I} \dim_{\overline{\mathbb{F}}_q} \text{Hom}_Q(P_i^{[-1]}, M) \alpha_i \\ &= \sum_{i \in I} \dim_{\overline{\mathbb{F}}_q} \text{Hom}_Q(P_{\sigma^{-1}(i)}, M) \alpha_i = \sum_{i \in I} \dim_{\overline{\mathbb{F}}_q} \text{Hom}_Q(P_i, M) \alpha_{\sigma(i)} = \sigma(\underline{\dim} M). \square \end{aligned}$$

Proposition 3.12. *Taking dimension vector defines a bijective correspondence between the set $\{SP_i \mid i \in I^\sigma\} \cup \text{ind mod}(\mathbb{F}_q Q^\sigma)$ and $\Phi_{\geq -1}(\Gamma^\sigma)$, namely, between $\text{ind}\mathcal{C}_{Q^\sigma}$ and the set of cluster variables, with \tilde{M}_α^F corresponding to $\tilde{\alpha}$ for any $\alpha \in \Phi_{\geq -1}(\Gamma)$.*

Proof. It follows from Lemma 3.11 and Proposition 3.4. \square

Let $\alpha, \beta \in \Phi_{\geq -1}(\Gamma)$, and M_α, M_β the corresponding indecomposable object of \mathcal{C}_Q as in Theorem 2.3. Let $d(\alpha)$ and $d(\beta)$ be the σ -period of α and β respectively (also F -period of M_α and M_β respectively). Then

$$(\tilde{\alpha} \parallel \tilde{\beta})_{\Gamma^\sigma} = \sum_{t=0}^{d(\beta)-1} (\alpha \parallel \sigma^t \beta)_\Gamma = \sum_{t=0}^{d(\beta)-1} \dim_{\overline{\mathbb{F}}_q} \text{Ext}_{\mathcal{C}_Q}^1(M_\alpha, M_\beta^{[1]}) = \dim_{\overline{\mathbb{F}}_q} \text{Ext}_{\mathcal{C}_Q}^1(M_\alpha, \tilde{M}_\beta).$$

Since $\dim_{\overline{\mathbb{F}}_q} \text{Ext}_{\mathcal{C}_Q}^1(M_\alpha, \tilde{M}_\beta) = \dim_{\overline{\mathbb{F}}_q} \text{Ext}_{\mathcal{C}_Q}^1(M_\alpha^{[r]}, \tilde{M}_\beta^{[r]}) = \dim_{\overline{\mathbb{F}}_q} \text{Ext}_{\mathcal{C}_Q}^1(M_\alpha^{[r]}, \tilde{M}_\beta)$ for any $r \in \mathbb{Z}$, it follows that

$$(\tilde{\alpha} \parallel \tilde{\beta})_{\Gamma^\sigma} = \frac{1}{d(\alpha)} \dim_{\overline{\mathbb{F}}_q} \text{Ext}_{\mathcal{C}_Q}^1(\tilde{M}_\alpha, \tilde{M}_\beta) = \frac{1}{d(\alpha)} \dim_{\mathbb{F}_q} \text{Ext}_{\mathcal{C}_{Q^\sigma}}^1(\tilde{M}_\alpha^F, \tilde{M}_\beta^F).$$

By 3.5 we have $D_{\tilde{M}_\alpha^F} = \mathbb{F}_{q^{d(\alpha)}}$, and hence

$$(\tilde{\alpha} \parallel \tilde{\beta})_{\Gamma^\sigma} = \dim_{\mathbb{F}_{q^{d(\alpha)}}} \text{Ext}_{\mathcal{C}_{Q^\sigma}}^1(\tilde{M}_\alpha^F, \tilde{M}_\beta^F) = \dim_{D_{\tilde{M}_\alpha^F}} \text{Ext}_{\mathcal{C}_{Q^\sigma}}^1(\tilde{M}_\alpha^F, \tilde{M}_\beta^F).$$

Therefore,

Theorem 3.13. (i) $(\tilde{\alpha} \parallel \tilde{\beta})_{\Gamma^\sigma} = \dim_{D_{M_\alpha^F}} \text{Ext}_{\mathcal{C}_{Q^\sigma}}^1(\tilde{M}_\alpha^F, \tilde{M}_\beta^F)$.

(ii) The bijective correspondence in Proposition 3.12 induces a one-to-one correspondence between isoclasses of tilting objects in \mathcal{C}_{Q^σ} and clusters.

(iii) Two elements $\tilde{\alpha}, \tilde{\beta}$ in $\Phi_{\geq -1}(\Gamma^\sigma)$ form an exchange pair if and only their compatibility degree is 1, i.e. $(\tilde{\alpha} \parallel \tilde{\beta})_{\Gamma^\sigma} = 1 = (\tilde{\beta} \parallel \tilde{\alpha})_{\Gamma^\sigma}$.

Proof. (ii) follows from (i). (iii) follows from (i) and (ii). \square

We point out that these results are not new. Proposition 3.12 and Theorem 3.13(i)(ii) are proved in [Z]. Proposition 3.12 is also a direct consequence of non-simply-laced version of Gabriel's theorem (see [DR]). As mentioned before Theorem 3.13(iii) is proved in [FZ2].

Let $\underline{x} = (x_{\tilde{k}})_{\tilde{k} \in I^\sigma}$ be a cluster corresponding to the tilting object $T = \oplus_{\tilde{k} \in I^\sigma} T_{\tilde{k}}$. Let $\tilde{k} \in I^\sigma$. Let $T_{\tilde{k}}^*$ be the other complement to the almost complete tilting object $\overline{T} = \oplus_{\tilde{i} \neq \tilde{k}} T_{\tilde{i}}$. Then $x'_{\tilde{k}} = \underline{\dim} T_{\tilde{k}}^*$ is the unique element in $\Phi_{\geq -1}(\Gamma^\sigma)$ different from $x_{\tilde{k}}$ such that $(\underline{x} \setminus \{x_{\tilde{k}}\}) \cup \{x'_{\tilde{k}}\}$ is again a cluster.

Let $(\underline{x}, B) = (\{-\tilde{\alpha}_i | i = 1, \dots, n\}, B_Q)$ be the initial seed and correspondingly we fix the tilting seed $(S\mathbb{F}_q Q^\sigma, B_Q)$. For a seed $(\underline{x}', B') = \mu_{\tilde{k}_t} \cdots \mu_{\tilde{k}_1}(\underline{x}, B)$, define $\phi(\underline{x}', B') = \delta_{\tilde{k}_t} \cdots \delta_{\tilde{k}_1}(S\mathbb{F}_q Q^\sigma, B_Q)$.

Theorem 3.14. ϕ is well-defined, i.e. ϕ does not depend on the choice of the sequence (k_1, \dots, k_t) . Moreover, ϕ is a bijection from the set of seeds to the set of tilting seeds such that for any $\tilde{k} \in I^\sigma$, we have a commutative diagram

$$\begin{array}{ccc} (\underline{x}', B') & \xrightarrow{\phi} & (T', B_{T'}) \\ \downarrow \mu_{\tilde{k}} & & \downarrow \delta_{\tilde{k}} \\ (\underline{x}'', B'') & \xrightarrow{\phi} & (T'', B_{T''}) \end{array}$$

Proof. By induction it follows from Proposition 3.7 and Theorem 3.13. \square

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